Math 3235 Probability Theory

4/11/23

X_i are i.i.d. Bernoulli

$p = 0.5$

$$Y_i = 2\left(X_i - \frac{1}{2}\right)$$

$Y_i = 1 \quad p = 0.5$

$Y_i = -1 \quad p = 0.5$

$$\mathbb{E}(Y_i) = 0$$

$$\mathrm{Var}(Y_i) = 1$$

$$\sum_{i=0}^{N} Y_i = 2\left(\sum_{i=1}^{N} X_i - N/2\right) = Z_N$$

p.m.f. $\mathrm{bin}\left(\left(z + \frac{N}{2}\right)/2,\ N,\ 0.5\right)$

$\mathrm{bin}(x, N, 0.5)$ p.m.f. N, 0.5

$$\mathbb{E}(Z_N) = 0$$

$$\mathrm{Var}(Z_N) = N$$

$$\frac{Z_N}{\sqrt{N}} = T_N$$

$$\mathbb{E}(T_N) = 0 \qquad \mathrm{Var}(T_N) = 1$$

0

$$Y_i = \begin{cases} 1 & p = 0.5 \\ -1 & p = 0.5 \end{cases}$$

$$T_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i$$

∞

$$Z_N = \sum_{i=1}^{N} Y_i$$

$$\mathbb{P}(Z_N = m) = \binom{N}{\frac{N}{2} + \frac{m}{2}} 2^{-N}$$

$$m = t\sqrt{N}$$

$$\frac{N! \; 2^{-N}}{\left(\frac{N}{2}+\frac{m}{2}\right)! \left(\frac{N}{2}-\frac{m}{2}\right)!}$$

$$N! \simeq \sqrt{2\pi} \sqrt{N} \, N^{N} e^{-N}$$

$$\frac{\sqrt{2\pi} \sqrt{N} \, N^{N} \, 2^{-N}}{2\pi \sqrt{\frac{N}{2}+\frac{m}{2}} \sqrt{\frac{N}{2}-\frac{m}{2}} \left(\frac{N}{2}-\frac{m}{2}\right)^{\frac{N}{2}-\frac{m}{2}} \left(\frac{N}{2}+\frac{m}{2}\right)^{\frac{N}{2}+\frac{m}{2}}}$$

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{N}}{\sqrt{\frac{N}{2}+\frac{m}{2}} \sqrt{\frac{N}{2}-\frac{m}{2}}} \frac{N^{N} \, 2^{-N}}{\left(\frac{N}{2}\right)^{\frac{N}{2}-\frac{m}{2}} \left(1-\frac{m}{N}\right)^{\frac{N}{2}-\frac{m}{2}} \left(\frac{N}{2}\right)^{\frac{N}{2}+\frac{m}{2}} \left(1+\frac{m}{N}\right)^{\frac{N}{2}+\frac{m}{2}}}$$

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{N}}{\sqrt{\frac{N}{2}+\frac{m}{2}} \sqrt{\frac{N}{2}-\frac{m}{2}}} \left(1-\frac{m}{N}\right)^{-\left(\frac{N}{2}-\frac{m}{2}\right)} \left(1+\frac{m}{N}\right)^{-\left(\frac{N}{2}+\frac{m}{2}\right)}$$

$$m = \tau \sqrt{N}$$

$$\frac{1}{\sqrt{2\pi}} \xrightarrow[N \to \infty]{} \frac{1}{\sqrt{2\pi}}$$

$$\frac{\sqrt{N}}{\sqrt{\frac{N}{2}-\frac{\tau\sqrt{N}}{2}} \sqrt{\frac{N}{2}+\frac{\tau\sqrt{N}}{2}}} \xrightarrow[N \to \infty]{} \frac{2}{\sqrt{N}}$$

$$\left(1-\frac{\tau}{\sqrt{N}}\right)^{-\left(\frac{N}{2}-\frac{\tau\sqrt{N}}{2}\right)}\left(1+\frac{\tau}{\sqrt{N}}\right)^{-\left(\frac{N}{2}+\frac{\tau\sqrt{N}}{2}\right)} =$$

$$\exp\left(-N\left(\left(\frac{1}{2}-\frac{\tau}{2\sqrt{N}}\right)\ln\left(1-\frac{\tau}{\sqrt{N}}\right)+\left(\frac{1}{2}+\frac{\tau}{2\sqrt{N}}\right)\ln\left(1+\frac{\tau}{\sqrt{N}}\right)\right)\right)$$

$$\ln(1+x) = x - \frac{x^2}{2}$$

$$e^{\left(-N\left(\frac{1}{2}-\frac{\tau}{2\sqrt{N}}\right)\left(-\frac{\tau}{\sqrt{N}}-\frac{\tau^2}{N}\right)+\left(\frac{1}{2}+\frac{\tau}{2\sqrt{N}}\right)\left(\frac{\tau}{\sqrt{N}}-\frac{\tau^2}{N}\right)+\dots\right)}$$

$$e^{-\frac{\tau^2}{2}+O\left(\frac{1}{N^{1/2}}\right)} \longrightarrow e^{-\tau^2/2}$$

$$\frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, \frac{2}{\sqrt{N}} \to dt$$

Conclusion:

$$\sqrt{N}\, T_N \to N(0,1)$$

Definition: we say that a sequence X_N converges in distribution to X if

$\forall x$,

$$\lim_{N\to\infty} \mathbb{P}(X_N \leq x) = \mathbb{P}(X \leq x)$$

$$X_N \Rightarrow X$$

X_0 uniform $[-1, 1]$

$$X_N = (-1)^N X_0$$

$$X_N \Rightarrow X_0$$

X_N is uniform in $[-1, 1]$

$$\mathbb{P}(X_N \leq x) = \mathbb{P}(X_0 \leq x) \quad \forall N$$

$$X_N \overset{\mathbb{P}}{\not\to} X_0$$

$$X_N - X_0 = \begin{cases} 0 & N \text{ even} \\ -2X_0 & N \text{ odd} \end{cases}$$

Convergence in distribution is very weak!

C.L.T.: if X_i are i.i.d. rv with $\mathbb{E}(X_i) = \mu$, $\text{var}(X_i) = \sigma^2$,

$$Z_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma}$$

Then

$$Z_N \Rightarrow \mathcal{N}(0,1)$$

$$\lim_{N \to \infty} \mathbb{P}(Z_N \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy$$

probability integral